

Second Order Phase Transition in Anisotropic Lattice Gauge Theories with Extra Dimensions

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Field theories with extra dimensions live in a limbo. While their classical solutions have been the subject of considerable study, their quantum aspects are difficult to control. A special class of such theories are anisotropic gauge theories. The anisotropy was originally introduced to localize chiral fermions. Their continuum limit is of practical interest and it will be shown that the anisotropy of the gauge couplings plays a crucial role in opening the phase diagram of the theory to a new phase, that is separated from the others by a second order phase transition. The mechanism behind this is generic for a certain class of models, that can be studied with lattice techniques. This leads to new perspectives for the study of quantum effects of extra dimensions.

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Extra dimensions¹ lead a troubled existence in field theory, since quantum field theories in more than four dimensions are plagued by untameable ultraviolet divergences, rendering any calculation, beyond the solution of the equations of motion, sensitive to the details of the regularization used. The lattice regularization has the advantage that it can be used to probe perturbative as well as non-perturbative aspects and thus provide hints about this sensitivity. We will use it to study a particular class of theories, anisotropic gauge theories [1] (with compact gauge group). We want to see what effect the anisotropy has on the order of the transitions between the various phases (so we specialize to the case of compact $U(1)$). While there has been a fair amount of numerical work [2, 3, 4, 5], the reason *why* a second order phase transition should, indeed, appear, has not been really spelled out. So it's useful to see how this could happen in a concrete example—as well as what could prevent its appearance. In addition, the method used has a wider applicability and deserves being recalled.

The action is in Wilson form

$$S = \beta \sum_n \sum_{1 \leq \mu < \nu \leq d_{\parallel}} (1 - \text{Re} [U_{\mu\nu}]) + \beta' \sum_n \sum_{d_{\parallel}+1 \leq \mu < \nu \leq d_{\parallel}+d_{\perp}} (1 - \text{Re} [U_{\mu\nu}]) \quad (1)$$

corresponding to the situation illustrated in fig. 1 for $d_{\parallel} = 2$ and $d_{\perp} = 1$. We shall use a technique

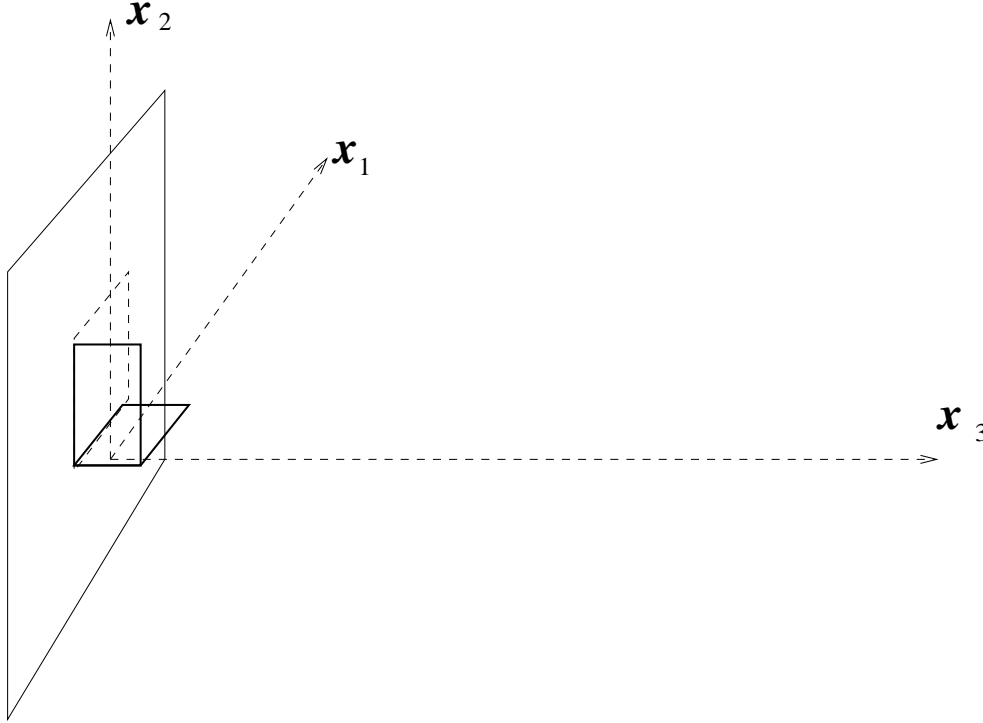


Figure 1: An example in $D = d_{\parallel} + d_{\perp} = 3$, with $d_{\parallel} = 2$ and $d_{\perp} = 1$. The plaquettes in the (x_2, x_3) and (x_3, x_1) planes enter with coefficient β' in the action; the plaquettes in the (x_1, x_2) plane enter with coefficient β .

developed for implementing the mean field approximation to systems with local symmetries to obtain the phase diagram, namely a trick introduced in ref. [6] (we use it in the form presented in [1, 7]).

¹More than the three spatial dimensions we typically perceive. We stick to one time dimension.

We insert in the partition function,

$$Z[J] = \int [\mathcal{D}U] e^{-S[U] + \sum_n \text{Re}(J_\mu(n) U_\mu(n))} \quad (2)$$

the expression

$$1 = \int \left[\prod_{\text{links}} \int d\text{Re}(V_l) d\text{Im}(V_l) \delta(\text{Re}(V_l) - \text{Re}(U_\mu(n))) \delta(\text{Im}(V_l) - \text{Im}(U_\mu(n))) \right] \quad (3)$$

to decouple the gauge links

$$\begin{aligned} Z[J] &= \int [\mathcal{D}U] \left[\prod_{\text{links}} \int d\text{Re}(V_l) d\text{Im}(V_l) \delta(\text{Re}(V_l) - \text{Re}(U_\mu(n))) \delta(\text{Im}(V_l) - \text{Im}(U_\mu(n))) \right] e^{-S[U] + \sum_n \text{Re}(J_\mu(n) U_\mu(n))} = \\ &= \int [\mathcal{D}U] \left[\prod_{\text{links}} \int d\text{Re}(V_l) d\text{Im}(V_l) \frac{d\alpha_l^R}{2\pi} \frac{d\alpha_l^I}{2\pi} e^{i \sum_l \alpha_l^R (-\text{Re}(V_l) + \text{Re}(U_\mu(n)))} e^{i \sum_l \alpha_l^I (-\text{Im}(V_l) + \text{Im}(U_\mu(n)))} \right] e^{-S[U] + \sum_n \text{Re}(J_\mu(n) U_\mu(n))} = \\ &= \int \left[\prod_{\text{links}} \int d\text{Re}(V_l) d\text{Im}(V_l) \frac{d\alpha_l^R}{2\pi} \frac{d\alpha_l^I}{2\pi} \right] e^{-S[\text{Re}(V_l), \text{Im}(V_l)] + \sum_l (\text{Re}(J_l) \text{Re}(V_l) - \text{Im}(J_l) \text{Im}(V_l)) - i \sum_l \alpha_l^R \text{Re}(V_l) - i \sum_l \alpha_l^I \text{Im}(V_l) + \sum_l w(\alpha_l^R, \alpha_l^I)} \end{aligned} \quad (4)$$

where $w(\alpha_l^R, \alpha_l^I)$ contains the information about the gauge group,

$$e^{w(\alpha_l^R, \alpha_l^I)} \equiv \int \mathcal{D}U e^{i(\alpha_l^R \text{Re}(U_\mu) + \alpha_l^I \text{Im}(U_\mu))} \quad (5)$$

So far we have an exact transcription: we have traded the *constrained* variables, $U_\mu(n)$ (that must satisfy $[\text{Re}(U_\mu(n))]^2 + [\text{Im}(U_\mu(n))]^2 = 1$), for the *unconstrained* variables, $\alpha_l^R, \alpha_l^I, \text{Re}(V_l), \text{Im}(V_l)$. It is, indeed, the existence of the constraint that leads to a non-trivial dependence on the coupling constant(s) of the effective action thus obtained, already at the “classical” level.

The effective action seems to have acquired terms that are complex—however the way they enter allows us to perform a “Wick rotation”, $i\alpha_l^R \equiv \hat{\alpha}_l^R, i\alpha_l^I \equiv \hat{\alpha}_l^I$ and obtain an action that is manifestly real:

$$S_{\text{eff}}(\hat{\alpha}_l^R, \hat{\alpha}_l^I, \text{Re}(V_l), \text{Im}(V_l)) = S[\text{Re}(V_l), \text{Im}(V_l)] + \sum_l (\hat{\alpha}_l^R \text{Re}(V_l) + \hat{\alpha}_l^I \text{Im}(V_l)) - \sum_l w(\hat{\alpha}_l^R, \hat{\alpha}_l^I) \quad (6)$$

We can, in fact, use this action for Monte Carlo simulations—but, also, for analytical computations, that are much easier to perform, since we have solved the constraints [6, 7, 8].

We now specialize eq. 6 to the case of the action in eq. (1) and look for extrema that are uniform along the d_{\parallel} —dimensional respectively along the d_{\perp} extra dimensions: $V_l \equiv v$, for links that belong in the d_{\parallel} —dimensional subspaces and $V_l \equiv v'$ for links that “point out” along the d_{\perp} extra dimensions. Similarly $\hat{\alpha}_l \equiv \hat{\alpha}$ within the d_{\parallel} dimensional subspaces and $\hat{\alpha}_l \equiv \hat{\alpha}'$ along the d_{\perp} extra dimensions. A plaquette that lies in the d_{\parallel} —dimensional subspace makes the following contribution to the effective action

$$\text{Re}[U_{\mu\nu}(n)]|_{1 \leq \mu < \nu \leq d_{\parallel}} = \text{Re}[(v^R + iv^I)(v^R + iv^I)(v^R - iv^I)(v^R - iv^I)] = ([v^R]^2 + [v^I]^2)^2 \quad (7)$$

Similarly, a plaquette that lies in the d_{\perp} —dimensional subspace contributes the expression

$$\text{Re}[U_{\mu\nu}(n)]|_{d_{\text{parallel}} + 1 \leq \mu < \nu \leq d_{\parallel} + d_{\perp}} = ([v'^R]^2 + [v'^I]^2)^2 \quad (8)$$

A plaquette that “spans” the subspace between two d_{\parallel} -dimensional subspaces contributes

$$\text{Re}[U_{\mu\nu}(n)]|_{1 \leq \mu \leq d_{\parallel} < \nu \leq d_{\parallel} + d_{\perp}} = \text{Re}((v^R + iv^I)(v'^R + iv'^I)(v^R - iv^I)(v'^R - iv'^I)) = ([v^R]^2 + [v^I]^2)([v'^R]^2 + [v'^I]^2) \quad (9)$$

Simple counting allows us to write down the expression for the effective action for such uniform configurations:

$$S_{\text{eff}}[v^R, v^I, v'^R, v'^I, \hat{\alpha}^R, \hat{\alpha}^I, \hat{\alpha}'^R, \hat{\alpha}'^I] = \beta \frac{d_{\parallel}(d_{\parallel} - 1)}{2} (1 - ([v^R]^2 + [v^I]^2)^2) + \beta' \frac{d_{\perp}(d_{\perp} - 1)}{2} (1 - ([v'^R]^2 + [v'^I]^2)^2) + \beta' d_{\perp} d_{\parallel} (1 - ([v^R]^2 + [v^I]^2)([v'^R]^2 + [v'^I]^2)) + d_{\parallel} (\hat{\alpha}^R v^R + \hat{\alpha}^I v^I - w(\hat{\alpha}^R, \hat{\alpha}^I)) + d_{\perp} (\hat{\alpha}'^R v'^R + \hat{\alpha}'^I v'^I - w(\hat{\alpha}'^R, \hat{\alpha}'^I)) \quad (10)$$

For the case of compact $U(1)$ the gauge group integral is given in terms of elementary functions:

$$e^{w(\hat{\alpha}^R, \hat{\alpha}^I)} = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{\hat{\alpha}^R \cos \theta + \hat{\alpha}^I \sin \theta} = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{\sqrt{[\hat{\alpha}^R]^2 + [\hat{\alpha}^I]^2} \cos(\theta - \phi_{\hat{\alpha}})} \equiv I_0 \left(\sqrt{[\hat{\alpha}^R]^2 + [\hat{\alpha}^I]^2} \right) \quad (11)$$

where $I_0(\cdot)$ is the modified Bessel function.

We notice that the group integral depends only on the length of the “vector(s)” $(\hat{\alpha}^R, \hat{\alpha}^I)$ —and that the plaquette terms in the effective action depend only on the length of the “vector(s)” (v^R, v^I) . The two vectors are coupled only through their “scalar product”, $\hat{\alpha}^R v^R + \hat{\alpha}^I v^I$, which depends on their lengths and their *relative* orientation. This means that we can *choose* a convenient coordinate system in this space and, as long as the corresponding symmetry isn’t spontaneously broken, we can simplify the calculations considerably. We thus choose the orientations so that $v^I = 0$, $v'^I = 0$, $\hat{\alpha}^I = 0$, $\hat{\alpha}'^I = 0$. Indeed we easily check that this choice is a solution of the equations for the extrema of the effective action. In a sense this amounts to “choosing a gauge” in this theory. To simplify notation we henceforth set $v^R \equiv v$, $v'^R \equiv v'$, $\hat{\alpha}^R \equiv \hat{\alpha}$, $\hat{\alpha}'^R \equiv \hat{\alpha}'$.

In this “gauge”, therefore, the action takes the form

$$S_{\text{eff}}[v, v', \hat{\alpha}, \hat{\alpha}'] = \beta \frac{d_{\parallel}(d_{\parallel} - 1)}{2} (1 - v^4) + \beta' \frac{d_{\perp}(d_{\perp} - 1)}{2} (1 - v'^4) + \beta' d_{\parallel} d_{\perp} (1 - v^2 v'^2) + d_{\parallel} (\hat{\alpha} v - w(\hat{\alpha})) + d_{\perp} (\hat{\alpha}' v' - w(\hat{\alpha}')) \quad (12)$$

Compactness of the gauge group implies that $w(0) = 1$ and $\infty > w''(0) > 0$. In addition, $w'(0) = 0$. These features may be seen to hold for compact $U(1)$ —but they hold for *any* compact group.

The extrema of the effective action are solutions of the equations

$$\begin{aligned} v &= dw(\hat{\alpha})/d\hat{\alpha} \\ v' &= dw(\hat{\alpha}')/d\hat{\alpha}' \\ \hat{\alpha} &= 2\beta d_{\parallel}(d_{\parallel} - 1)v^3 + 2\beta' d_{\parallel} d_{\perp} v v'^2 \\ \hat{\alpha}' &= 2\beta' d_{\perp}(d_{\perp} - 1)v'^3 + 2\beta' d_{\parallel} v^2 v' \end{aligned} \quad (13)$$

These equations always possess the solution $(\hat{\alpha}, \hat{\alpha}', v, v') = (0, 0, 0, 0)$ that corresponds to the confining phase—the string tension is infinite. However they also have non-zero solutions, that depend on the values of the couplings β and β' . The reason this is possible is that uniform configurations are only invariant under global (constant) gauge transformations—and Elitzur’s theorem [9] holds

only if local transformations are possible. Thus it is *not* a contradiction of Elitzur's theorem but rather a consequence of the fact that the assumption behind it does not hold for the configuration under study.

We thus find a solution with $(\hat{\alpha}, \hat{\alpha}', v, v') \neq (0, 0, 0, 0)$, which corresponds to a $d_{\parallel} + d_{\perp}$ -dimensional Coulomb phase (since Wilson loops with perimeter $L = L_1 + L_2$ behave as v^L , v'^L or $v^{L_1} v'^{L_2}$).

However there also exists a solution with $\hat{\alpha} \neq 0, v \neq 0, \hat{\alpha}' = 0, v' = 0$. In this phase (named the “layered phase” in ref. [1]) the Wilson loops show perimeter behavior within a d_{\parallel} -dimensional subspace (since $v \neq 0$) and show confinement along the d_{\perp} directions, since $v' = 0$. There isn't any “bulk” at all: the $d_{\parallel} + d_{\perp}$ -dimensional space has become a stack of d_{\parallel} -dimensional layers. Since the string tension is infinite the layers are infinitely thin and the theory on them is local. Corrections to the mean field approximation will make this string tension finite—the layers will acquire a thickness, inversely proportional to the (square root of the) string tension and the theory will display *non-local* features, if probed at such length scales. For this to be consistent this string tension should be much larger than the tension of the fundamental string.

In all cases considered here the boundary conditions are assumed to be periodic, but all dimensions are assumed to become infinite in the continuum limit.

It is interesting to try and see whether the transition from one phase to another can become continuous. Indeed the mean field approximation to lattice gauge theories typically predicts first order (discontinuous transitions). The reason can be understood from the expression of the action: the plaquette terms, in the isotropic case, are quartic in the link variables. The only terms that can contribute to quadratic order are the “constraint” terms, $\hat{\alpha}v - w(\hat{\alpha})$. If we replace $v = dw(\hat{\alpha})/d\hat{\alpha}$ and expand to quadratic order, around $\alpha = 0$, we find that this point corresponds to a minimum of the effective action, that can never become a maximum. Therefore, if another minimum appears for $\alpha \neq 0$, the transition is, necessarily, of first order. Such a minimum, corresponding to a Coulomb phase, is only credible for a theory with a $U(1)$ factor: the putative Coulomb phase turns out to be an artefact of the mean field approximation [8] for Yang–Mills theories with a simple Lie group and Monte Carlo simulations indicate that they are always confining at strong coupling and asymptotically free at weak coupling [10].

In the case under study here, however, there is a term in the action that *can* destabilize the confining phase in a way consistent with a continuous transition: the term

$$S_{\text{eff}}^{\text{mixed}} = \beta' d_{\parallel} d_{\perp} (1 - v^2 v'^2) \quad (14)$$

is quadratic in the link variables, due to the anisotropy. And these variables enter with a sign that allows them to destabilize the confining phase along the d_{\perp} directions. To see this we expand the effective action around the solution $(\hat{\alpha}, \hat{\alpha}' = 0)$, which exists for β large enough and β' small enough, within the subspace where $v = dw(\hat{\alpha})/d\hat{\alpha}$ and $v' = dw(\hat{\alpha}')/d\hat{\alpha}'$. So we consider α' small enough that we may expand around $\hat{\alpha}' = 0$ to quadratic order—but we retain the exact dependence on α . We find

$$S_{\text{eff}}[v, v', \hat{\alpha}, \hat{\alpha}'] \approx S_{\text{eff}}[v, 0, \hat{\alpha}, 0] + \hat{\alpha}'^2 w''(0) d_{\perp} \left[-\beta' d_{\parallel} v^2(\hat{\alpha}) w''(0) + \frac{1}{2} \right] \quad (15)$$

This expression depends on β implicitly, since $\hat{\alpha} = \hat{\alpha}(\beta)$. If $v(\hat{\alpha}) \neq 0$ —the system is in the Coulomb

phase within a d_{\parallel} -dimensional subspace—there is a line,

$$\beta'_{\text{crit}}(\beta) = \frac{1}{2d_{\parallel}v^2(\hat{\alpha})w''(0)} \quad (16)$$

such that, for $\beta' < \beta'_{\text{crit}}$ the system is in the layered phase and for $\beta' > \beta'_{\text{crit}}$ it is in the $d_{\parallel} + d_{\perp}$ -dimensional Coulomb phase through a continuous transition. For $U(1)$, in particular, $w''(0) = 1/2$ and $v(\hat{\alpha})$ is a bounded function of $\hat{\alpha}(\beta)$, that tends to 1 as $\hat{\alpha}(\beta) \rightarrow \infty$. In that limit, which is relevant as $\beta \rightarrow \infty$, we obtain that $\beta'_{\text{crit}} \rightarrow 1/d_{\parallel}$, a result that is compatible with the mean field approximation, which may be considered an expansion in $1/d_{\parallel}$ (and was found in another way in ref. [1]). This has further interesting consequences since, many years ago, Peskin [11] noted that at a second order phase transition point the static quark–anti-quark potential, derived from the Wilson loop, would display $1/R$ behavior independently of the dimensionality. To date an example of such a system was not available. Anisotropic lattice gauge theories with a $U(1)$ factor could provide such an example and it will be interesting to explore its consequences further through Monte Carlo simulation.

In conclusion we have identified the mechanism that is behind the transition from the layered phase to the bulk Coulomb phase for anisotropic lattice gauge theories, whose symmetry group contains a $U(1)$ factor. Monte Carlo simulations to check its validity beyond the mean field approximation are certainly called for and the theory in the continuum limit, especially in the presence of matter, remains to be constructed. Theories with $U(N) \sim U(1) \times SU(N)$ symmetry group have been studied in four dimensions [12] and the special behavior of the $U(1)$ factor had been remarked upon—it would be most interesting to study quantitatively the role of the anisotropy. One expects the $U(1)$ factor to trigger the appearance of the layered phase [13].

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